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Perfect Measures, Nuclear Spaces and the Convex Compactness Property

Medidas Perfectas, Espacios Nucleares y la Propiedad de Compacidad Convexa

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Abstract

It is proved that for certain kinds of K-spaces X, the spaces $(C_b(X,E),\beta_p)$ has the convex compactness property if E is a Banach space. Also, if X is a real-compact K-spaces then $(C_b(X,E),\beta_p)$ is a nuclear space if and only if X is finite and E is finite dimensional.

Key words and phrases: P-spaces, K-spaces, Do-spaces, real-compact spaces, convex compactness property, nuclear spaces.

Resumen

Se prueba que para ciertos tipos de K-espacios X, los espacios $(C_b(X,E),\beta_p)$ tienen la propiedad de compacidad convexa si E es un espacio de Banach. También, si X es un K-espacio real-compacto, entonces $(C_b(X,E),\beta_p)$ es un espacio nuclear si y solo si X es finito y E es finito dimensional.

Palabras y frases clave: P-espacios, K-espacios, Do-espacios, espacios real-compactos, propiedad de compacidad convexa, espacios nucleares.

1 Introduction

Let X be a completely regular Hausdorff space, E a Banach space. By $C_b(X)$ we will denote the set of all bounded real-valued continuous function on X and $C_b(X,E)$ denotes all bounded continuous functions from X into E. $C_b(X)\otimes E$ denotes the tensor product of $C_b(X)$ and E [5]. Sentilles in [6] defined locally convex topologies β_0 and β_1 on $C_b(X)$, which yield the spaces of $M_t(X)$ and $M_\sigma(X)$ of tight and σ- additives Baire measures on X as dual spaces. Koumoullis in [4] defined a new topology β_p on $C_b(X)$, and redefined the topology β_∞ on $C_b(X)$ which yield the spaces $M_p(X)$ and $M_\infty(X)$ of perfect and uniform Baire measure on X as dual space. For the vector case see [2],[3],[8].

Let us recall that a completely regular Hausdorff space X is called a *K-space* if it has the weak topology determined by the family of its compact subsets, that is to say that a set $A\subseteq X$ is *closed* iff $A\cap K$ is closed for all compact subsets K of X. A locally convex space is said to have

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the *convex compactness property* if for every compact K its closed absolutely convex hull is also compact. The easiest way for a locally convex space E to have the convex compactness property is that of being complete or quasicomplete in the Mackey topology [5]. A spaces X is said to be a D_0-*space* if its real-compactification νX and its topological completion θX coincide. If F is a locally convex space and $B \neq \emptyset$ is a convex, circled and bounded subset of F, then $F_1 = \bigcup_{n=1}^{\infty} n\ B$ is a subspaces of F. The gauge function P_B of B in F_1 is easily seen to be a norm on F_1. The normed space (F_1, P_B) is denoted by F_B. A linear map $u : E \to F$ is said to be *nuclear* if it is of the form

$$x \to u(x) = \sum_{n=1}^{\infty} \quad \lambda_n f_n(x) y_n$$

where $\sum_{n=1}^{\infty} \mid \lambda_n \mid < \infty$, $\{f_n\}$ is an equicontinuous sequence in E' and $\{y_n\}$ is a sequence contained in a convex, circled and bounded subset B of F for which F_B is complete. A locally convex space E is said to be *nuclear* if every continuous linear map of E into any Banach space is nuclear.

2 Nuclearity and the Convex Compactness Property

Theorem 1. *Let X be a K-space and a D_0-space, E a Banach space and $H \subseteq C_b(X, E)$. Then, the following conditions are equivalent:*

(*a*) *H is uniformly bounded, equicontinuous, and $H(x)$ is relatively compact in E for every $x \in X$.*

(*b*) *H is β_p-relatively compact.*

(*c*) *H is β_p-precompact.*

Proof. We see $(a) \Rightarrow (b)$. Suppose that H is a uniformly bounded, equicontinuous subset of $C_b(X, E)$ such that $H(x)$ is relatively compact subset of E for every $x \in X$. Then, the pointwise closure $\overline{H}$ of H is also equicontinuous and by Ascoli's Theorem it is precompact in the compact-open topology on $C_b(X, E)$. Now, since $\overline{H}$ is uniformly bounded, we have that on $\overline{H}$, β_0 is the compact-open topology and so, $\overline{H}$ is also β_0-precompact. Also, since X is a K-space, it is know that $(C_b(X, E), \beta_0)$ is complete ([1]). Then in this case, $\overline{H}$ will also be β_0-compact and since β_∞ is the finest locally convex topology agreeing with the pointwise topology on the uniformly bounded subsets of $C_b(X, E)$, we have that $\overline{H}$ will also be β_∞-compact. But, we have that X is a D_0-space, then $\beta_p \leq \beta_\infty$ implies that $\overline{H}$ is β_p-compact so H is relatively β_p-compact.

$(b) \Rightarrow (c)$ is trivial. Finally, we see $(c) \Rightarrow (a)$. Suppose then that H is β_p-precompact. Since $\beta_0 \leq \beta_p$ it follows that H is also β_0-precompact then H is β_0-bounded which also implies that H is uniformly bounded. Now, since β_0 is the finest locally convex topology agreeing with the compact-open topology on uniformly bounded subset of $C_b(X, E)$, it follows that H is precompact respect to the compact open topology, then by Ascoli's theorem H when restricted to each compact subset of X is equicontinuous. But X is a K-space then it follows that H is equicontinuous. Now, let $x \in X$ and we prove that $H(x)$ is relatively compact. Since H is pre-compact in the pointwise topology, every net $\{f_\alpha\}$ in H has a Cauchy subnet $\{f_\beta\}$. Therefore, $\{f_\beta(x)\}$ is Cauchy in E for every $x \in X$. Then the result follows. □

Theorem 2. *Let X be a K-space and a D_0-space and E a Banach space. Then, $(C_b(X,E),\beta_p)$ has the convex compactness property.*

Proof. Let A be a β_p-compact subset of $C_b(X,E)$. Then, the absolutely convex hull of A will be β_p-precompact and by Theorem 1, the closed absolutely convex hull of A will be β_p-compact. □

Theorem 3. *Let X be a realcompact K-space, the $(C_b(X),\beta_p)$ is a nuclear space if and only if X is finite.*

Proof. Clearly if X is finite, then $(C_b(X),\beta p)$ is topologically isomorphic to $\mathbb{R}^n$, being n the cardinality of X. Now, since $\mathbb{R}^n$ is nuclear, the conclusion follows. Now let us suppose that $(C_p(X),\beta_p)$ is nuclear, then every bounded subset is β_p-precompact ([5]), thus the closed unit ball $B=\{f\in C_b(X):\|f\|\leq 1\}$ is β_p-precompact. Now, since every realcompact space is topologically complete, by Theorem 1, B is β_0-compact and since $\beta_0\leq\beta_p$ we have that B is β_o-compact which implies that X is discrete ([7]). Then, X is a realcompact metric space. But then $M_p(X)=M_t(X)$ and since X is a P-space, we have that both β_0 and β_p are Mackey's topologies ([1], [3]), and so, $\beta_0=\beta_p$ and $(C_b(X),\beta_0)$ is a nuclear space, which implies that X is finite ([1]). □

Theorem 4. *Let X be realcompact K-space and E a Banach normed space. Then, $(C_b(X,E),\beta_p)$ is a nuclear space if and only if X is finite and E is finite dimensional.*

Proof. If X is finite and E is finite dimensional then, as in the proof of Theorem 3, $(C_b(X,E),\beta_p)$ is topologically isomorphic to E^n for some n. Then, $(C_b(X,E),\beta_p)$ is a nuclear space. Conversely, suppose that $(C_b(X,E),\beta_p)$ is nuclear. For a fixed $e\in E$, we have that $(C_b(X),\beta_p)$ is topologically isomorphic to the subspace $C_b(X)\otimes e$ of $(C_b(X,E),\beta_p)$. Since every subspace of a nuclear space is again nuclear, we get that $(C_b(X),\beta_p)$ is nuclear, and by Theorem 3, X is finite. Also, we know that E is embedded as a subspace of $(C_b(X,E),\beta_p)$ and so, E is a normed nuclear space, then E is finite dimensional. □

Theorem 5. *If X is a K-space and E is Banach space then $(C_b(X,E),\beta_p)$ is sequentially complete.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $(C_b(X,E),\beta_p)$. Since X is a K-space, $(C_b(X,E),\beta_0)$ is complete ([1]) and since $\beta_0\leq\beta_p$ we have that $\{f_n\}$ is β_0-convergent to function $f\in C_b(X,E)$. We claim that $\{f_n\}$ also converges to f in $(C_b(X,E),\beta_p)$. In fact, since β_p has a base W of β_p-closed absolutely convex sets which are weakly closed and since $\mid\mu\mid(\|f_n-f\|)\to 0$ by the Dominated Converge Theorem, if $U\in W$ then there exist $N_0\geq 1$ integer such that for every $n\geq N_0$,

$$\mid\mu(f_n)-\mu(f)\mid\leq\mid\mu\mid(\|f_n-f\|)\quad \mu\in M_p(X)$$

then $f_n-f\in U$ and the theorem holds. □

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